Approximation Algorithms for NP-Complete Problems

Algorithms: Design and Analysis, Part II

Analysis of a Greedy Knapsack Heuristic

Performance Guarantee

Theorem: Value of the 3-step greedy algorithm's solution is always $\geq 50\%$· value of an optimal solution.

Thought experiment: What if we were allowed to fill fully the knapsack using a suitable "fraction" (like 70%) of item (k+1)? [The value of which is "pro-rated"]

$\Rightarrow$ Will call this the "greedy fractional solution"

Example: $W = 3$, $v_1 = 3$, $v_2 = 2$, $w_1 = w_2 = 2$

get 100% get 50%

$\Rightarrow$ Greedy fractional solution has value 4

Quiz

Question: Let F = value of greedy fractional solution and OPT = value of optimal (non-fractional) solution. Which of the following is true?

A) $F = OPT$ for every knapsack instance

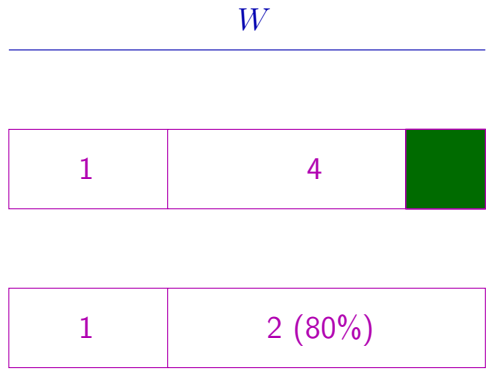
B) $F > OPT$ for every knapsack instance

C) $F \leq OPT$ for every instance, and can be strict

C) $F \geq OPT$ for every instance, and can be strict

Proof Sketch

Claim: Greedy fractional solution at least as good as every non-fractional feasible solution.



(1) Let S = an arbitrary feasible solution
(2) Suppose l units of knapsack filled by S with items not packed by the greedy fractional solution
(3) Must be at least l units of knapsack filled by greedy fractional solution not packed by S
(4) By greedy criterion, items in (3) have larger bang-per-buck v_i/w_i than those in (2) [i.e., more valuable use of space]
(5) Total value of greedy fractional solution at least that of S

Analysis of Greedy Heuristic

In Step 2, suppose our greedy algorithm picks the 1st k items (sorted by v_i/w_i).

by step 3

Value of 3-step greedy algorithm $\geq$ total value of 1st k items

also is $\geq$ value of $(k+1)$th item

$\Rightarrow$ 2·(value of 3-step greedy) $\geq$ total value of 1st $(k+1)$ items

$\geq$ total value of greedy fractional soln

$\geq$ optimal knapsack solution

QED!

Analysis is Tight

Example: $W = 1000$

$v_1 = 502$ $\quad v_2 = v_3 = 500$

$w_1 = 501$ $\quad w_2 = w_3 = 500$

$\Rightarrow$ 3-step greedy solution has value 502

$\Rightarrow$ optimal solution has value 1000

A Refined Analysis

Suppose: Every item i has size $w_i \leq 10\%$·knapsack capacity W.

Consequence: If greedy algorithm fails to pack all items in Step 2, then the knapsack is $\geq 90\%$ full.

$\Rightarrow$ Value of 2-step greedy algorithm
$\geq 90\%$· value of greedy fractional solution
$\geq 90\%$· value of an optimal solution.

[In general, if $\max_i w_i \leq \delta W$, then 2-step greedy value is $\geq (1-\delta)$·optimal]