Advanced Union-Find

Union by Rank - Analysis

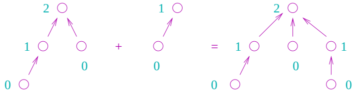
Algorithms: Design and Analysis, Part II

Properties of Ranks

Recall: Lazy Unions.

Invariant (for now): rank[x] $=$ max $\#$ of hops from a leaf to x.
[Note $\max_x$rank[x] $\approx$ worst-case running time of FIND.]

Union by Rank: Make old root with smaller rank child of the root with the larger rank.
[Choose new root arbitrarily in case of a tie, and add 1 to its rank.]



Immediate from Invariant/Rank Maintenance:

(1) For all objects x, rank[x] only goes up over time

(2) Only ranks of roots can go up
[once x a non-root, rank[x] frozen forevermore]

(3) Ranks strictly increase along a path to the root

Rank Lemma

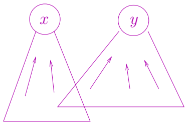
Rank Lemma: Consider an arbitrary sequence of UNION (+FIND) operations. For every $r \in \{0, 1, 2, \ldots\}$, there are at most $n/2^r$ objects with rank r.

Corollary: Max rank always $\leq \log_2 n$

Corollary: Worst-case running time of FIND, UNION is $O(\log n)$.
[With Union by Rank.]

Proof of Rank Lemma

Claim 1: If x, y have the same rank r, then their subtrees (objects from which can reach x, y) are disjoint.



Claim 2: The subtree of a rank-r object has size $\geq 2^r$.
[Note Claim 1 + Claim 2 imply the Rank Lemma.]

Proof of Claim 1: Will show contrapositive. Suppose subtrees of x, y have object z in common $\Rightarrow \exists$ paths $z \rightarrow x, z \rightarrow y$
$\Rightarrow$ One of x, y is an ancestor of the other
$\Rightarrow$ The ancestor has strictly larger rank. [By property (3)]
QED (Claim 1)

Proof of Claim 2

Rank $r \Rightarrow$ Subtree size $\geq 2^r$

Base case: Initially all ranks $= 0$, all subtree sizes $= 1$

Inductive step: Nothing to prove unless the rank of some object changes (subtree sizes only go up).

Interesting case: UNION(x, y), with s_1=FIND(x), s_2=FIND(y), and rank$[s_1]$=rank$[s_2]$=$r \Rightarrow s_2$'s new rank $= r + 1$
$\Rightarrow s_2$'s new subtree size $= s_2$'s old subtree size $+ s_1$'s old subtree size (each at least 2^r by the inductive hypothesis) $\geq 2^{r+1}$. QED!