Approximation Algorithms for NP-Complete Problems

Analysis of a Dynamic Programming Heuristic for Knapsack

Algorithms: Design and Analysis, Part II

The Dynamic Programming Heuristic

Step 1: Set $\hat{v}_i = \lfloor \frac{v_i}{m} \rfloor$ for every item i.

Step 2: Compute optimal solution with respect to the $\hat{v}_i$'s using dynamic programming.

Plan for analysis:

(1) Figure out how big we can take m, subject to achieving a $(1-\epsilon)$-approximation

(2) Plug in this value of m to determine running time

Quiz

Question: Suppose we round v_i to the value $\hat{v}_i$. Which of the following is true?

A) $\hat{v}_i$ is between $v_i - m$ and v_i

B) $\hat{v}_i$ is between v_i and $v_i + m$

C) $m\hat{v}_i$ is between $v_i - m$ and v_i

D) $m\hat{v}_i$ is between $v_i - m$ and v_i

Accuracy Analysis I

From quiz: Since we rounded down to the nearest multiple of m, $m\hat{v}_i \in [v_i - m, v_i]$ for each item i.

Thus: (1) $v_i \geq m\hat{v}_i$, (2) $m\hat{v}_i \geq v_i - m$

Also: If $S^* =$ optimal solution to the original problem (with the original v_i's), and $S =$ our heuristic's solution, then

$$(3) \quad \sum_{i \in S} \hat{v}_i \geq \sum_{i \in S^*} \hat{v}_i$$

[Since S is optimal for the $\hat{v}_i$'s] (recall Step 2)

Accuracy Analysis II

S = our solution, S^*= optimal solution

$$\sum_{i \in S} v_i \overset{(1)}{\geq} m \sum_{i \in S} \hat{v}_i \overset{(3)}{\geq} m \sum_{i \in S^*} \hat{v}_i \overset{(2)}{\geq} \sum_{i \in S^*} (v_i - m)$$

contains at most n items

Thus: $\sum_{i \in S} v_i \geq (\sum_{i \in S^*} v_i) - mn$

Constraint: $\sum_{i \in S} v_i \geq (1 - \epsilon) \sum_{i \in S^*} v_i$

To achieve above constraint: Choose m small enough that $mn \leq \epsilon \sum_{i \in S^*} v_i$

unknown to algorithm, but definitely $\geq v_{\max}$

Sufficient: Set m so that $mn = \epsilon v_{\max}$, i.e., heuristic uses $m = \frac{\epsilon v_{\max}}{n}$

Running Time Analysis

Point: Setting $m = \frac{\epsilon v_{\max}}{n}$ guarantees that value of our solution is $\geq (1-\epsilon)\cdot$value of optimal solution.

Recall: Running time is $O(n^2 \hat{v}_{\max})$

Note: For every item i, $\hat{v}_i \leq \frac{v_i}{m} \leq \frac{v_{\max}}{m} = v_{\max} \frac{n}{\epsilon v_{\max}} = \frac{n}{\epsilon}$

Running time $= O(n^3/\epsilon)$