Dynamic Programming

Algorithms: Design and Analysis, Part II

Optimal BSTs: Optimal Substructure

Problem Definition

Input: Frequencies $p_1, p_2, \ldots, p_n$ for items $1, 2, \ldots, n$.
[Assume items in sorted order, $1 < 2 < \ldots < n$]

Goal: Compute a valid search tree that minimizes the weighted (average) search time.

$C(T) = \sum_{\text{items } i} p_i$ [search time for i in T]

Depth of i in $T + 1$

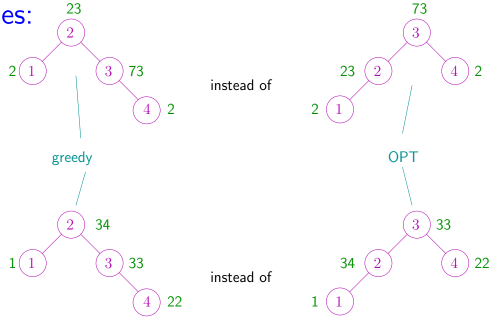
Greedy Doesn't Work

Intuition: Want the most (respectively, least) frequently accessed items closest (respectively, furthest) from the root.

Ideas for greedy algorithms:

- Bottom-up [populate lowest level with least frequently accessed keys]
- Top-down [put most frequently accessed item at root, recurse]

Counter examples:



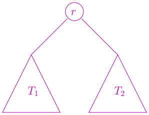
Choosing the Root

Issue: With the top-down approach, the choice of root has hard-to-predict repercussions further down the tree.
[stymies both greedy and naive divide + conquer approaches]

Idea: What if we knew the root?
(i.e., maybe can try all possibilities within a dynamic programming algorithm!)

Optimal Substructure

Question: Suppose an optimal BST for keys $\{1, 2, \ldots, n\}$ has root r, left subtree T_1, right subtree T_2. Pick the strongest statement that you suspect is true.



A) Neither T_1 nor T_2 need be optimal for the items it contains.

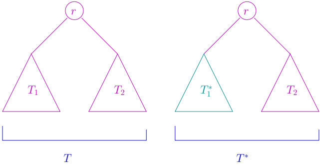
B) At least one of T_1, T_2 is optimal for the items it contains.

C) Each of T_1, T_2 is optimal for the items it contains.

D) <u>T_1 is optimal for the keys $\{1, 2, \ldots, r-1\}$ and T_2 for the keys $\{r+1, r+2, \ldots, n\}$</u>

Proof of Optimal Substructure

Let T be an optimal BST for keys $\{1, 2, \ldots, n\}$ with frequencies $p_1, \ldots, p_n$. Suppose T has root r. Suppose for contradiction that T_1 is not optimal for $\{1, 2, \ldots, r-1\}$ [other case is similar] with $C(T_1^*) < C(T_1)$. Obtain T^* from T by "cutting+pasting" T_1^* in for T_1.



Note: To complete contradiction + proof, only need to show that $C(T^*) < C(T)$.

Proof of Optimal Substructure (con'd)

A Calculation:

$=1+$search time for i in T_1 $\qquad$ $=1+$search time for i in T_2

$$
\begin{aligned}
C(T) &= \sum_{i=1}^{n} p_i \text{ [search time for } i \text{ in } T] \\
&= p_r \cdot 1 + \sum_{i=1}^{r-1} p_i \text{ [search time for } i \text{ in } T] \\
&\quad + \sum_{i=r+1}^{n} p_i \text{ [search time for } i \text{ in } T] \\
&= \sum_{i=1}^{n} p_i + \sum_{i=1}^{r-1} p_i \text{ [search time for } i \text{ in } T_1] \\
&\quad + \sum_{i=r+1}^{n} p_i \text{ [search time for } i \text{ in } T_2]
\end{aligned}
$$

a constant (independent of T) $\qquad = C(T_1) \qquad = C(T_2)$

Similarly: $C(T^*) = \sum_{i=1}^{n} p_i + C(T_1^*) + C(T_2)$

Upshot: $C(T_1^*) < C(T_1)$ implies $C(T^*) < C(T)$, contradicting optimality of T. QED!