The Bellman-Ford Algorithm

Optimal Substructure

Algorithms: Design and Analysis, Part II

Single-Source Shortest Path Problem, Revisited

Input: Directed graph $G = (V, E)$, edge lengths c_e [possibly negative], source vertex $s \in V$.

Goal: either

(A) For all destinations $v \in V$, compute the length of a shortest s-v path $\rightarrow$ focus of this + next video

OR

(B) Output a negative cycle (excuse for failing to compute shortest paths) $\rightarrow$ later

Optimal Substructure (Informal)

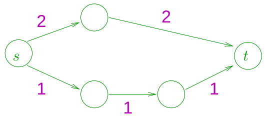
Intuition: Exploit sequential nature of paths. Subpath of a shortest path should itself be shortest.

Issue: Not clear how to define "smaller" & "larger" subproblems.

Key idea: Artificially restrict the number of edges in a path.

Subproblem size $\Longleftrightarrow$ Number of permitted edges

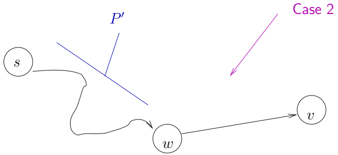
Example:



Optimal Substructure (Formal)

Lemma: Let $G = (V, E)$ be a directed graph with edge lengths c_e and source vertex s.
[G might or might not have a negative cycle]
For every $v \in V$, $i \in \{1, 2, \ldots\}$, let P = shortest s-v path with at most i edges. (Cycles are permitted.)

Case 1: If P has $\leq (i-1)$ edges, it is a shortest s-v path with $\leq (i-1)$ edges.
Case 2: If P has i edges with last hop (w, v), then P' is a shortest s-w path with $\leq (i-1)$ edges.



Proof of Optimal Substructure

Case 1: By (obvious) contradiction.

Case 2: If Q (from s to w, $\leq (i-1)$ edges) is shorter than P' then $Q + (w, v)$ (from s to v, $\leq i$ edges) is shorter than $P' + (w, v)$ $(= P)$ which contradicts the optimality of P. QED!

Quiz

Question: How many candidates are there for an optimal solution to a subproblem involving the destination v?

A) 2

B) $\boxed{1 + \text{in-degree}(v)}$

C) $n - 1$

D) n

1 from Case 1 + 1 from Case 2 $\underline{\text{for each choice of the final hop } (w, c)}$