Minimum Spanning Trees

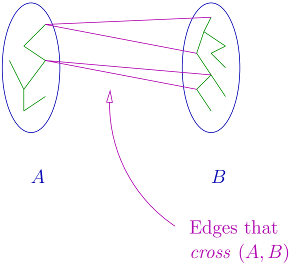
Algorithms: Design and Analysis, Part II

Correctness of Prim's Algorithm (Part I)

Cuts

Claim: Prim's algorithm outputs a spanning tree.

Definition: A cut of a graph $G = (V, E)$ is a partition of V into 2 non-empty sets.



Quiz on Cuts

Question: Roughly how many cuts does a graph with n vertices have?

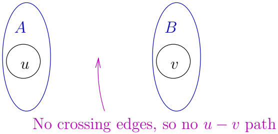
A) n C) 2^n (for each vertex, choose whether in A or in B)

B) n^2 D) n^n

Empty Cut Lemma

Empty Cut Lemma: A graph is not connected $\iff$ $\exists$ cut (A, B) with no crossing edges.

Proof: $(\Leftarrow)$ Assume the RHS. Pick any $u \in A$ and $v \in B$. Since no edges cross (A, B) there is no u, v path in G. $\Rightarrow$ G not connected.

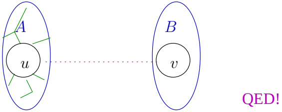


$(\Rightarrow)$ Assume the LHS. Suppose G has no $u - v$ path. Define

$A = \{$Vertices reachable from u in $G\}$ (u's connected component)

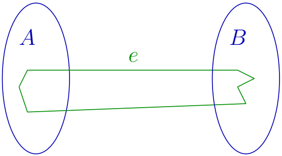
$B = \{$All other vertices$\}$ (all other connected components)

Note: No edges cross cut (A, B) (otherwise A would be bigger!)



Two Easy Facts

Double-Crossing Lemma: Suppose the cycle $C \subseteq E$ has an edge crossing the cut (A, B): then so does some other edge of C.

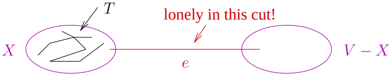


Lonely Cut Corollary: If e is the only edge crossing some cut (A, B), then it is not in any cycle. [If it were in a cycle, some other edge would have to cross the cut!]

Proof of Part I

Claim: Prim's algorithm outputs a spanning tree.
[Not claiming MST yet]

Proof: (1) Algorithm maintains invariant that T spans X
[straightforward induction - you check]



(2) Can't get stuck with $X \neq V$
[otherwise the cut $(X, V - X)$ must be empty; by Empty Cut Lemma input graph G is disconnected]

(3) No cycles ever get created in T. Why? Consider any iteration, with current sets X and T. Suppose e gets added.
Key point: e is the first edge crossing $(X, V - X)$ that gets added to $T \Rightarrow$ its addition can't create a cycle in T (by Lonely Cut Corollary). QED!