Minimum Spanning Trees

Proof of the Cut Property

Algorithms: Design and Analysis, Part II

The Cut Property

Assumption: Distinct edge costs.

CUT PROPERTY: Consider an edge e of G. Suppose there is a cut (A, B) such that e is the cheapest edge of G that crosses it. Then e belongs to the MST of G.

Proof Plan

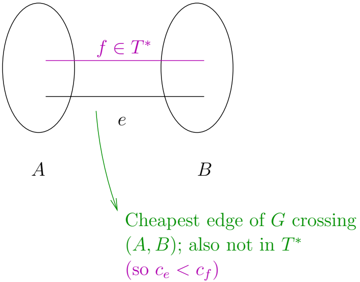
Will argue by contradiction, using an exchange argument.
[Compare to scheduling application]

Suppose there is an edge e that is the cheapest one crossing a cut (A, B), yet e is not in the MST T^*.

Idea: Exchange e with another edge in T^* to make it even cheaper (contradiction).

Question: Which edge to exchange e with?

Attempted Exchange



Note: Since T^* is connected, must construct an edge $f (\neq e)$ crossing (A, B).

Idea: Exchange e and f to get a spanning tree cheaper than T^* (contradiction).

Exchanging Edges

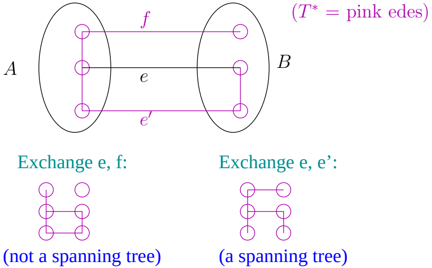
Question: Let T^* be a spanning tree of G, $e \notin T^*$, $f \in T^*$. Is $T^* \cup \{e\} - \{f\}$ a spanning tree of G?

A) Yes always

B) No never

C) If e is the cheapest edge crossing some cut, then yes

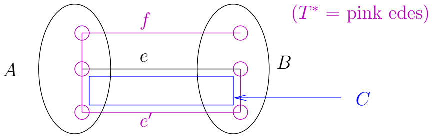
D) Maybe, maybe not (depending on the choice of e and f)



Smart Exchanges

Hope: Can always find suitable edge e' so that exchange yields bona fide spanning tree of G.

How? Let C = cycle created by adding e to T^*.



By the Double-Crossing Lemma: Some other edge e' of C [with $e' \neq e$ and $e' \in T^*$] crosses (A, B).

You check: $T = T^* \cup \{e\} - \{e'\}$ is also a spanning tree.

Since $c_e < c_{e'}$, T cheaper than purported MST T^*, contradiction.