Algorithms: Design and Analysis, Part II

Exact Algorithms for NP-Complete Problems

Smarter Search for Vertex Cover

The Vertex Cover Problem

Given: An undirected graph $G = (V, E)$.

Goal: Compute a minimum-cardinality vertex cover (a set $S \subseteq V$ that includes at least one endpoint of each edge of E).

Suppose: Given a positive integer k as input, we want to check whether or not there is a vertex cover with size $\leq k$. [Think of k as "small"]

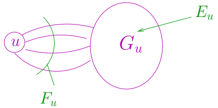
Note: Could try all possibilities, would take $\approx \binom{n}{k} = \Theta(n^k)$ time.

Question: Can we do better?

A Substructure Lemma

Substructure Lemma: Consider graph G, edge $(u, v) \in G$, integer $k \geq 1$. Let $G_u = G$ with u and its incident edges deleted (similarly, G_v). Then G has a vertex cover of size $k \iff G_u$ or G_v (or both) has a vertex cover of size $(k - 1)$

Proof: $(\Leftarrow)$ Suppose G_u (say) has a vertex cover S of size $k - 1$. Write $E = E_u$ (inside G_u) $\cup$ F_u (incident to u)



Since S has an endpoint of each edge of E_u, $S \cup \{u\}$ is a vertex cover (of size k) of G.

$(\Rightarrow)$ Let S = a vertex cover of G of size k. Since (u, v) an edge of G, at least one of u, v (say u) is in S. Since no edges of E_u incident on u, $S - \{u\}$ must be a vertex cover (of size $k - 1$) of G_u. QED!

A Search Algorithm

[Given undirected graph $G = (V, E)$, integer k]

[Ignore base cases]

(1) Pick an arbitrary edge $(u, v) \in E$.

(2) Recursively search for a vertex cover S of size $(k-1)$ in G_u (G with u + its incident edges deleted).
If found, return $S \cup \{u\}$.

(3) Recursively search for a vertex cover S of size $(k-1)$ in G_v.
If found, return $S \cup \{v\}$.

(4) FAIL. [G has no vertex cover with size k]

Analysis of Search Algorithm

Correctness: Straightforward induction, using the substructure lemma to justify the inductive step.

Running time: Total number of recursive calls is $O(2^k)$ [branching factor ≤ 2, recursion depth $\leq k$] (formally, proof by induction on k)

- Also, $O(m)$ work per recursive call (not counting work done by recursive subcalls)

$\Rightarrow$ Running time = $\boxed{O(2^k m)}$

Polynomial-time as long as $k = O(\log n)$

Remains feasible even when $k \approx 20$

Way better than $\Theta(n^k)$!