Advanced Union-Find

The Ackermann Function

Algorithms: Design and Analysis, Part II

Tarjan's Bound

Theorem: [Tarjan 75] With Union by Rank and path compression, m UNION+FIND operations take $O(m\alpha(n))$ time, where $\alpha(n)$ is the inverse Ackerman function (will define in this video)

Proof in next video.

The Ackermann Function

Aside: Many different definitions, all more or less equivalent.

Will define $A_k(r)$ for all integers k and $r \geq 1$. (recursively)

Base case: $A_0(r) = r + 1$ for all $r \geq 1$.

In general: For $k, r \geq 1$:

$$\begin{aligned} A_k(r) &= \text{Apply } A_{k-1} \ r \text{ times to } r \\ &= \boxed{(A_{k-1} \circ A_{k-1} \circ \ldots \circ A_{k-1})(r)} \end{aligned}$$

r-fold composition

Quiz: A_1

Quiz: $A_1(r)$ corresponds to what function of r?

A) Successor ($r \mapsto r + 1$)

B) Doubling ($r \mapsto 2r$)

C) Exponentation ($r \mapsto 2^r$)

D) Tower function ($r \mapsto 2^{2^{\cdots\ r \text{ times } \cdots^{2}}}$)

$A_1(r) = (A_0 \circ A_0 \circ \ldots \circ A_0)(r) = 2r$

(r-fold composition, add 1 each time)

Quiz: A_2

Quiz: What function does $A_2(r)$ correspond to?

A) $r \mapsto 4r$

B) $r \mapsto 2^r$

B) $\boxed{r \mapsto r2^r}$

D) $r \mapsto 2^{2^{\cdots^{r \text{ times}} \cdots^{2}}}$

$A_2(r) = (A_1 \circ A_1 \circ \ldots \circ A_1)(r) = r2^r$

(r-fold composition, doubles each time)

Quiz: A_3

Quiz: What is $A_3(2)$? Recall $A_2(r) = r2^r$

A) 8

B) 1024

B) $\boxed{2048}$

D) Bigger than 2048

$A_3(2) = A_2(A_2(2)) = A_2(8) = 82^8 = 2^{11} = 2048$

In general: $A_3(r) = (A_2 \circ A_2 \circ \ldots (r \text{ times}) \ldots \circ A_2)(r) \geq$ a tower of r 2's $= 2^{2^{\cdots\ r \text{ times}\ \cdots^2}}$

A_4

$A_4(2) = A_3(A_3(2)) = A_3(2048) \geq 2^{2^{\cdots \text{ height } 2048 \cdots^{2}}}$

In general: $A_4(r) = (A_3 \circ \ldots \; r \text{ times } \ldots \circ A_3)(r) \approx$ iterated tower function (aka "wowzer" function)

The Inverse Ackermann Function

Definition: For every $n \geq 4$, $\alpha(n) =$ minimum value of k such that $A_k(2) \geq n$.

$\alpha(n) = 1,\ n = 4\ (A_1(2) = 4)$

$\alpha(n) = 2,\ n = 5, \ldots, 8\ (A_2(2) = 8)$

$\alpha(n) = 3,\ n = 9, 10, \ldots, 2048$

$\alpha(n) = 4$, n up to roughly a tower of 2's of height 2048

$\alpha(n) = 5$ for n up to ???

$\log^* n = 1,\ n = \underline{2}$

$\log^* n = 2,\ n = 3, \underline{4}$

$\log^* n = 3,\ n = 5, \ldots, \underline{16}$

$\log^* n = 4,\ n = 17, \underline{65536}$

$\log^* n = 5,\ n = 65537, \underline{2^{65536}}$

$\log^* n = 2048$ for such n