All-Pairs Shortest Paths (APSP)

Algorithms: Design and Analysis, Part II

The Floyd-Warshall Algorithm

Quiz

Setup: Let A = 3-D array (indexed by i, j, k).

Intent: $A[i,j,k]$ = length of a shortest i-j path with all internal nodes in $\{1, 2, \ldots, k\}$ (or $+\infty$ if no such paths)

Question: What is $A[i,j,0]$ if

(1) $i = j$ (2)$(i,j) \in E$ (3) $i \neq j$ and $(i,j) \notin E$

A) 0, 0, and $+\infty$

B) 0, c_{ij}, and c_{ij}

C) 0, c_{ij}, and $+\infty$

D) $+\infty$, c_{ij}, and $+\infty$

The Floyd-Warshall Algorithm

Let A = 3-D array (indexed by i, j, k)

Base cases: For all $i, j \in V$:

$$A[i,j,0] = \left\{ \begin{array}{l} 0 \text{ if } i = j \\ c_{ij} \text{ if } (i,j) \in E \\ +\infty \text{ if } i \neq j \text{ and } (i,j) \notin E \end{array} \right\}$$

For $k = 1$ to n

For $i = 1$ to n

For $j = 1$ to n

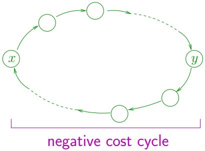
$$A[i,j,k] = \min \left\{ \begin{array}{ll} A[i,j,k-1] & \text{Case 1} \\ A[i,k,k-1] + A[k,j,k-1] & \text{Case 2} \end{array} \right\}$$

Correctness: From optimal substructure + induction, as usual.

Running time: $O(1)$ per subproblem, $O(n^3)$ overall.

Odds and Ends

Question #1: What if input graph G has a negative cycle?



Answer: Will have $A[i, i, n] < 0$ for at least one $i \in V$ at end of algorithm.

Question #2: How to reconstruct a shortest i-j path?

Answer: In addition to A, have Floyd-Warshall compute $B[i, j] =$ max label of an internal node on a shortest i-j path for all $i, j \in V$.

[Reset $B[i, j] = k$ if 2nd case of recurrence used to compute $A[i, j, k]$]

$\Rightarrow$ Can use the $B[i, j]$'s to recursively reconstruct shortest paths!