The Bellman-Ford Algorithm

Detecting Negative Cycles

Algorithms: Design and Analysis, Part II

Checking for a Negative Cycle

Question: What if the input graph G has a negative cycle?
[Want algorithm to report this fact]

Claim:

G has no negative-cost cycle (that is reachable from s) $\iff$ In the extended Bellman-Ford algorithm, $A[n-1, v] = A[n, v]$ for all $v \in V$.

Consequence: Can check for a negative cycle just by running Bellman-Ford for one extra iteration (running time still $O(mn)$).

Proof of Claim

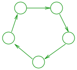
($\Rightarrow$) Already proved in correctness of Bellman-Ford
($\Leftarrow$) Assume $A[n-1,v] = A[n,v]$ for all $v \in V$. (Assume also these are finite ($< +\infty$))
Let $d(v)$ denote the common value of $A[n-1,v]$ and $A[n,v]$.

Recall algorithm:

$$\overbrace{A[n,v]}^{d(v)} = \min \left\{ \begin{array}{l} A[i-1,v] \\ \min_{(w,v)\in E}\{\overbrace{A[n-1,w]}^{d(w)} + c_{wv}\} \end{array} \right\}$$

Thus: $d(v) \le d(w) + c_{wv}$ for all edges $(w,v) \in E$

Equivalently: $d(v) - d(w) \le c_{wv}$

Now: Consider an arbitrary cycle C. 

$$\sum_{(w,v)\in C} \ge \sum_{(w,v)\in C}(d(w) - d(v)) = 0$$ QED!