The Bellman-Ford Algorithm

The Basic Algorithm

Algorithms: Design and Analysis, Part II

The Recurrence

Notation: Let $L_{i,v}$ = minimum length of a s-v path with $\leq i$ edges.

- With cycles allowed
- Defined as $+\infty$ if no s-v paths with $\leq i$ edges

Recurrence: For every $v \in V$, $i \in \{1, 2, \ldots\}$

$$L_{i,v} = \min \left\{ \begin{array}{ll} L_{(i-1),v} & \text{Case 1} \\ \min_{(u,v) \in E} \{L_{(i-1),w} + c_{wv}\} & \text{Case 2} \end{array} \right\}$$

Correctness: Brute-force search from the only (1+in-deg(v)) candidates (by the optimal substructure lemma).

If No Negative Cycles

Now: Suppose input graph G has no negative cycles.

$\Rightarrow$ Shortest paths do not have cycles

[removing a cycle only decreases length]

$\Rightarrow$ Have $\leq (n-1)$ edges

Point: If G has no negative cycle, only need to solve subproblems up to $i = n - 1$.

Subproblems: Compute $L_{i,v}$ for all $i \in \{0, 1, \ldots, n-1\}$ and all $v \in V$.

The Bellman-Ford Algorithm

Let A = 2-D array (indexed by i and v)

Base case: $A[0, s] = 0$; $A[0, v] = +\infty$ for all $v \neq s$.

For $i = 1, 2, \ldots, n-1$

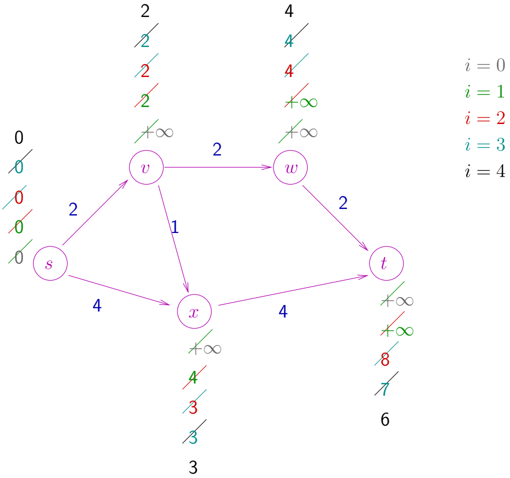
For each $v \in V$

$$A[i, v] = \min \left\{ \begin{array}{l} A[i-1, v] \\ \min_{(w,v) \in E} \{A[i-1, w] + c_{wv}\} \end{array} \right\}$$

As discussed: If G has no negative cycle, then algorithm is correct [with final answers $= A[n-1, v]$'s]

Example

$$A[i,v] = \min \left\{ \begin{array}{l} A[i-1,v] \\ \min_{(w,v)\in E}\{A[i-1,w] + c_{wv}\} \end{array} \right\}$$



Quiz

Question: What is the running time of the Bellman-Ford algorithm? [Pick the strongest true statement.] [$m = \#$ of edges, $n = \#$ of vertices]

A) $O(n^2)$ $\rightarrow$ # of subproblems, but might do $\Theta(n)$ work for one subproblem

B) $\boxed{O(mn)}$

C) $O(n^3)$

D) $O(m^2)$

Reason: Total work is $O(\boxed{n} \boxed{\sum_{v \in V} \text{in-deg}(v)}) = O(mn)$

iterations of outer loop (i.e. choices of i)

work done in one iteration $= m$

Stopping Early

Note: Suppose for some $j < n-1$, $A[j,v] = A[j-1,v]$ for all vertices v.

$\Rightarrow$ For all v, all future $A[i,v]$'s will be the same

$\Rightarrow$ Can safely halt (since $A[n-1,v]$'s $=$ correct shortest-path distances)