Local Search

Random Walks on a Line

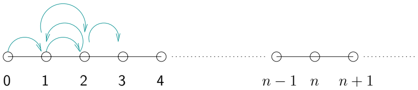
Algorithms: Design and Analysis, Part II

Random Walks

Key to analyzing Papadimitriou's algorithm:

Random walks on the nonnegative integers (trust me!)

Setup: Initially (at time 0), at position 0.



At each time step, your position goes up or down by 1, with 50/50 probability.

[Except if at position 0, in which case you move to position 1 with 100% probability]

Quiz

Notation: For an integer $n \geq 0$, let T_n = number of steps until random walk reaches position n.
[A random variable, sample space = coin flips at all time steps]

Question: What is $E[T_n]$? (your best guess)

A) $\Theta(n)$

B) $\boxed{\Theta(n^2)}$

C) $\Theta(n^3)$

D) $\Theta(2^n)$

Coming up: $E[T_n]=n^2$.

Analysis of T_n

Let Z_i = number of random walk steps to get to n from i. (Note $Z_0 = T_n$)

Edge cases: $E[Z_n]=0$, $E[Z_0]=1+E[Z_1]$

For $i \in \{1, 2, \ldots, n-1\}$

$$E[Z_i] = \overbrace{\Pr[\text{go left}]}^{1/2}\ \overbrace{E[Z_i \mid \text{go left}]}^{(1+E[Z_{i-1}])} + \overbrace{\Pr[\text{go right}]}^{1/2}\ \overbrace{E[Z_i \mid \text{go right}]}^{(1+E[Z_{i+1}])}$$

$$= 1 + \tfrac{1}{2}E[Z_{i+1}] + \tfrac{1}{2}E[Z_{i-1}]$$

Rearranging: $E[Z_i] - E[Z_{i+1}] = E[Z_{i-1}] - E[Z_i] + 2$

Finishing the Proof of Claim

So:

$$\mathsf{E}[Z_0]\text{-}\cancel{\mathsf{E}[Z_1]}=1$$

$$\cancel{\mathsf{E}[Z_1]}\text{-}\cancel{\mathsf{E}[Z_2]}=3$$

$$\cancel{\mathsf{E}[Z_2]}\text{-}\cancel{\mathsf{E}[Z_3]}=5$$

$$\vdots$$

$$+\quad \cancel{\mathsf{E}[Z_{n-1}]}\text{-}\cancelto{0}{\mathsf{E}[Z_n]}=2n-1$$

$\frac{n}{2}$ pairs of numbers, each sums to $2n$

$$\mathsf{E}[Z_0] = n^2$$

$$||$$

$$\mathsf{E}[T_n]$$

QED!

A Corollary

Corollary: $\Pr[T_n > 2n^2] \leq \frac{1}{2}$. (Special case of Markov's inequality)

Proof: Let p denote $\Pr[T_n > 2n^2]$.

$$\begin{aligned}
\text{We have } n^2 &= E[T_n] \\
\text{by last claim} \quad &= \underbrace{\sum_{k=0}^{2n^2} k\Pr[T_n = k]}_{\geq 0} + \sum_{k=2n^2+1}^{\infty} \underbrace{k}_{\geq 2n^2} \Pr[T_n = k] \\
&\geq 2n^2 \Pr[T_n > 2n^2] \\
&= 2n^2 p.
\end{aligned}$$

$\Rightarrow p \leq \frac{1}{2}$ QED!