



Algorithms: Design
and Analysis, Part II

Local Search

Random Walks on a
Line

Random Walks

Key to analyzing Papadimitriou's algorithm:

Random walks on the nonnegative integers (trust me!)

Setup: Initially (at time 0), at position 0.



At each time step, your position goes up or down by 1, with 50/50 probability.

[Except if at position 0, in which case you move to position 1 with 100% probability]

Quiz

Notation: For an integer $n \geq 0$, let T_n = number of steps until random walk reaches position n .

[A random variable, sample space = coin flips at all time steps]

Question: What is $E[T_n]$? (your best guess)

A) $\Theta(n)$

B) $\Theta(n^2)$

C) $\Theta(n^3)$

D) $\Theta(2^n)$

Coming up: $E[T_n] = n^2$.

Analysis of T_n

Let Z_i = number of random walk steps to get to n from i . (Note $Z_0 = T_n$)

Edge cases: $E[Z_n]=0$, $E[Z_0]=1+E[Z_1]$

For $i \in \{1, 2, \dots, n-1\}$

$$\begin{aligned} E[Z_i] &= \Pr[\text{go left}] E[Z_i \mid \text{go left}] + \Pr[\text{go right}] E[Z_i \mid \text{go right}] \\ &= 1 + \frac{1}{2} E[Z_{i+1}] + \frac{1}{2} E[Z_{i-1}] \end{aligned}$$

Diagram annotations: Arrows point from the text above to terms in the equation. From '1/2' to 'Pr[go left]'. From '(1+E[Z_{i-1}])' to 'E[Z_i | go left]'. From '1/2' to 'Pr[go right]'. From '(1+E[Z_{i+1}])' to 'E[Z_i | go right]'.

Rearranging: $E[Z_i] - E[Z_{i+1}] = E[Z_{i-1}] - E[Z_i] + 2$

Finishing the Proof of Claim

So:

$$\begin{array}{rcl} E[Z_0] - E[Z_1] & = & 1 \\ E[Z_1] - E[Z_2] & = & 3 \\ E[Z_2] - E[Z_3] & = & 5 \\ & \vdots & \\ + E[Z_{n-1}] - E[Z_n] & = & 2n - 1 \end{array}$$

Diagram: Purple curved arrows group the terms on the right side of the equations. The first arrow groups the 1 and 3, the second groups the 3 and 5, and the third groups the 5 and the ellipsis. A final arrow groups the last term, $2n - 1$. A purple text label points to these groupings: $\frac{n}{2}$ pairs of numbers, each sums to $2n$.

$$E[Z_0] = n^2$$

||

$$E[T_n]$$

QED!

A Corollary

Corollary: $\Pr[T_n > 2n^2] \leq \frac{1}{2}$. (Special case of Markov's inequality)

Proof: Let p denote $\Pr[T_n > 2n^2]$.

We have $n^2 = E[T_n]$

by last claim

$$= \sum_{k=0}^{2n^2} k \Pr[T_n = k] + \sum_{k=2n^2+1}^{\infty} k \Pr[T_n = k]$$

$$\geq 2n^2 \Pr[T_n > 2n^2]$$

$$= 2n^2 p.$$

$$\Rightarrow p \leq \frac{1}{2} \quad \text{QED!}$$