

# Equilibria, Computation, and Compromises

Some Points of Contact Between  
Theoretical Computer Science  
and Economics

Tim Roughgarden (Stanford)

# First Point of Contact

## Origins of game theory:

- "Zur Theorie der Gesellschaftsspiele" (1928)
- *Theory of Games and Economic Behavior* (1944, with Oskar Morgenstern)

## Early contributions to computing:

- ENIAC (UPenn, 1945)
- IAS machine (1945-1951)



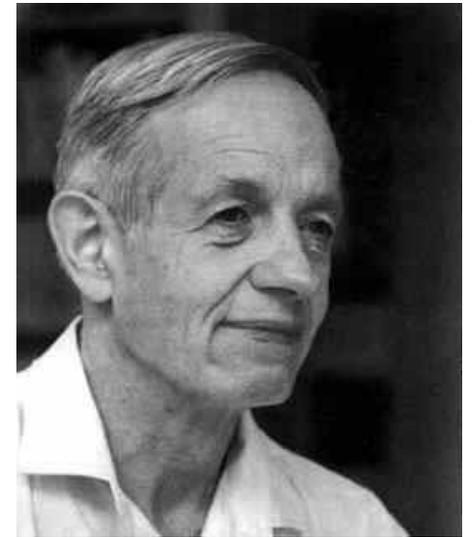
von Neumann

# Universal Existence

**Nash's Theorem (1950):** every finite noncooperative game has at least one (Nash) equilibrium.



An equilibrium



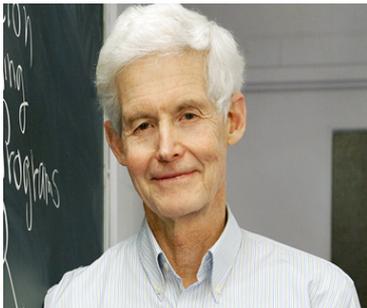
Nash

# NP-Completeness

## Cook-Karp-Levin Theorem (1971-1973):

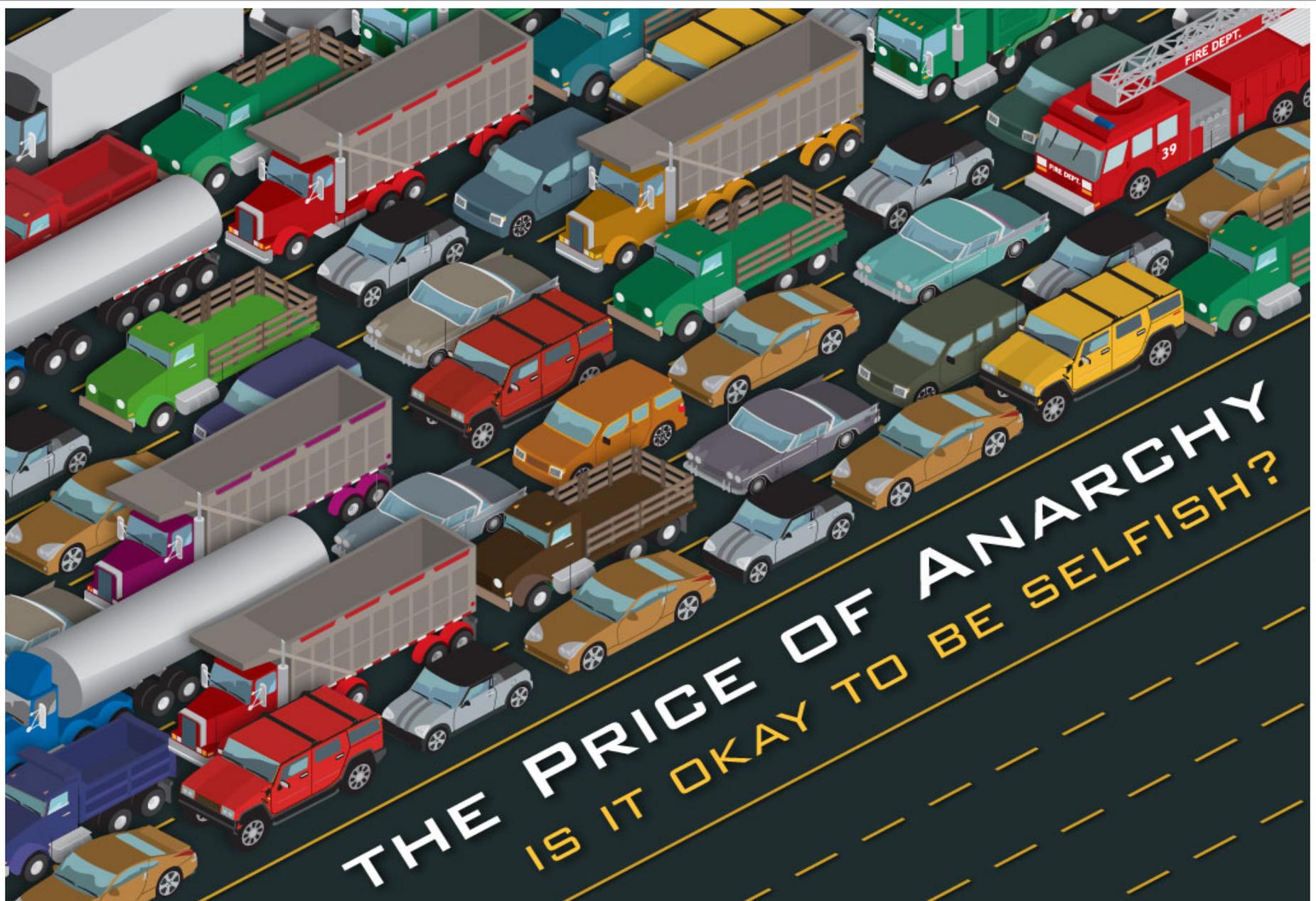
Many fundamental problems are “NP-complete”.

- “NP” = easy to verify an alleged solution
- “-complete” = difficult to determine if a solution exists
- compromises required (use heuristics, tackle special/small cases, buy lots of hardware, etc.)



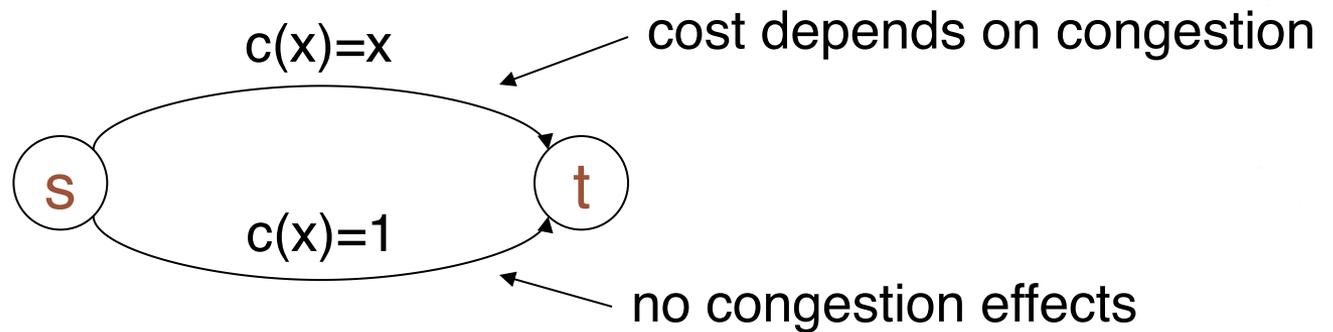
# Outline

1. Introduction: von Neumann, Nash, Cook-Karp-Levin
2. *Approximation: A Constructive Compromise*
3. Auction Design: The Rubber Meets the Road
4. Complexity: Bad Grades for the Nash Equilibrium
5. Conclusions



# Pigou's Example

**Example:** one unit of traffic wants to go from  $s$  to  $t$

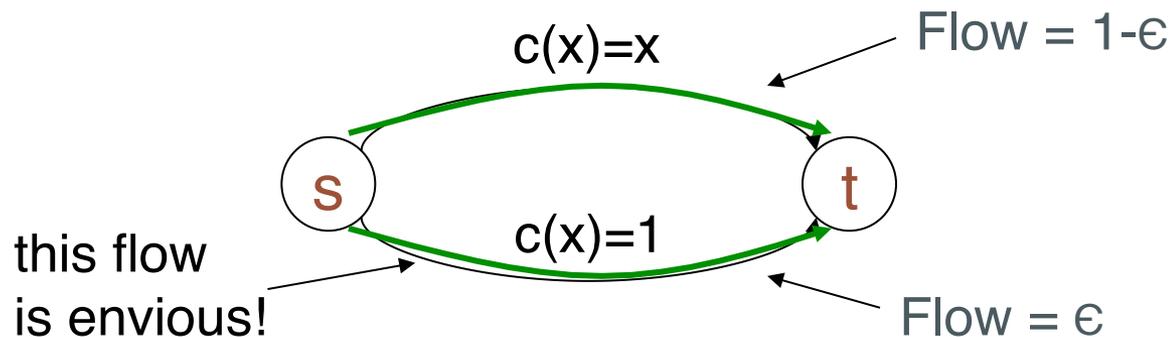


**Question:** what will selfish drivers do?

- assume everyone wants smallest-possible cost
- [Pigou 1920]

# Equilibrium in Pigou's Example

**Claim:** all traffic will take the top link.

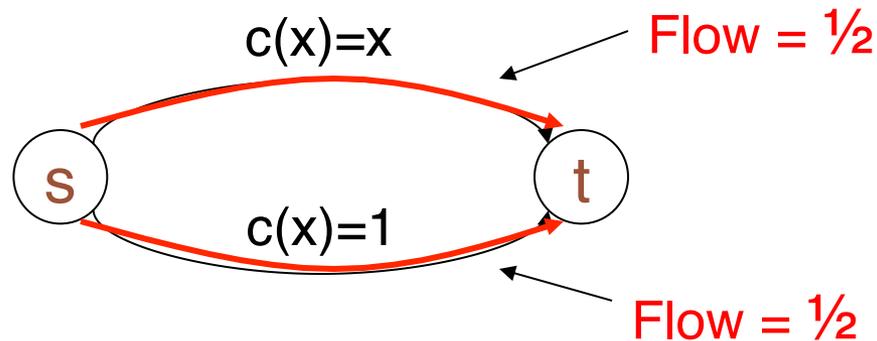


**Reason:**

- $\epsilon > 0 \Rightarrow$  traffic on bottom is envious
- $\epsilon = 0 \Rightarrow$  equilibrium
  - all traffic incurs one unit of cost

# Can We Do Better?

Consider instead: traffic split equally

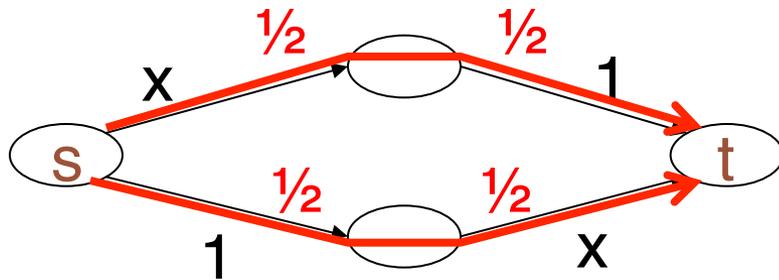


Improvement:

- half of traffic has cost 1 (same as before)
- half of traffic has cost  $1/2$  (much improved!)
- “price of anarchy” [Kousoupias/Papadimitriou 99] =  $4/3$

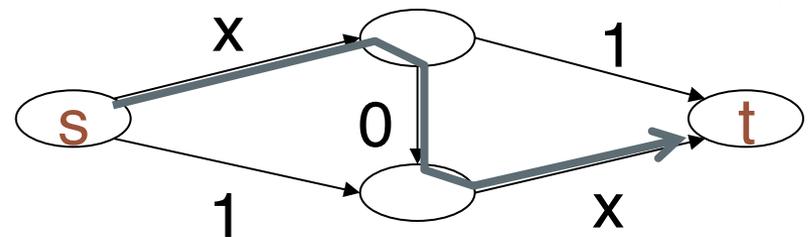
# Braess's Paradox

Initial Network:



Cost = 1.5

Augmented Network:

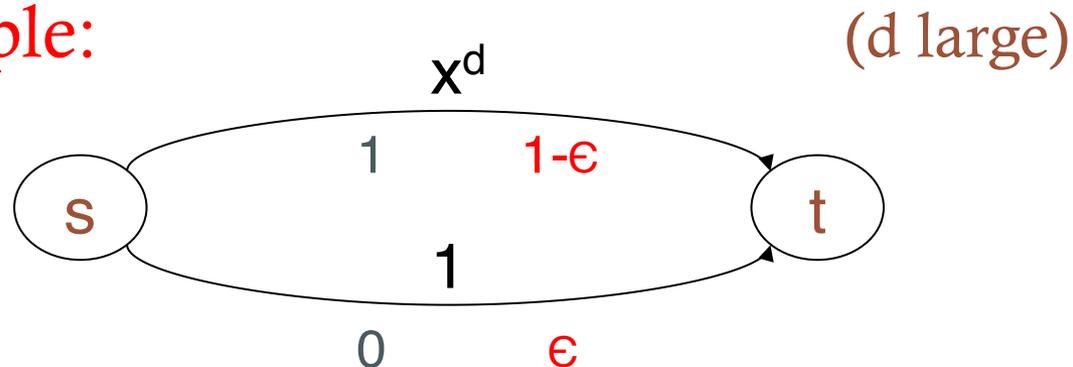


Cost = 2

Price of anarchy =  $\frac{4}{3}$  in augmented network (again!)

# A Nonlinear Pigou Network

Bad Example:



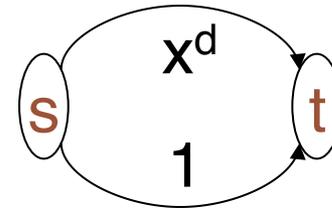
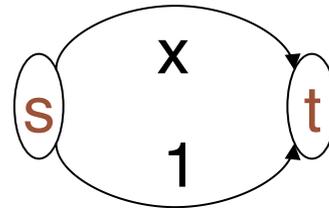
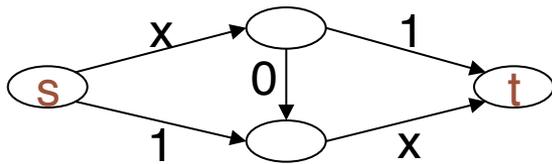
equilibrium has cost 1, **min cost  $\rightarrow 0$**

$\Rightarrow$  price of anarchy unbounded as  $d \rightarrow$  infinity

**Goal:** weakest-possible conditions under which the price of anarchy is small.

# When Is the Price of Anarchy Bounded?

Examples so far:



**Hope:** imposing additional structure on the cost functions helps

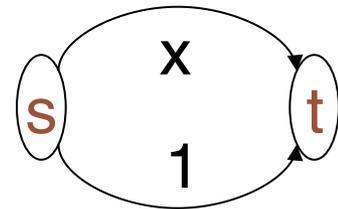
- worry: bad things happen in larger networks

# Affine Cost Functions

**Defn:** *affine cost function* has the form  $c_e(x) = a_e x + b_e$

**Theorem:** [Roughgarden/Tardos 00] for every network with affine cost functions:

$$\text{cost of eq flow} \leq 4/3 \times \text{cost of opt flow}$$



# Benefit of Overprovisioning

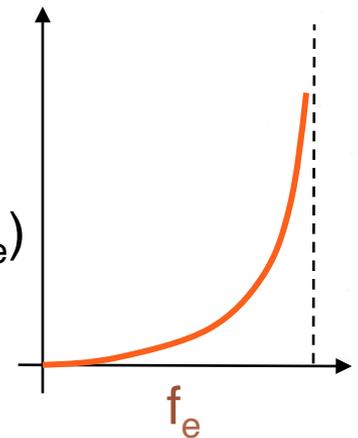
**M/M/1 Cost Functions:**  $c(x) = 1/(u_e - x)$

**Suppose:** network is overprovisioned by  $\beta > 0$   
(i.e.,  $\beta$  fraction of each edge unused).

**Then:** Price of anarchy is at most  $\frac{1}{2}(1 + 1/\sqrt{\beta})$

- arbitrary network size/topology, traffic matrix

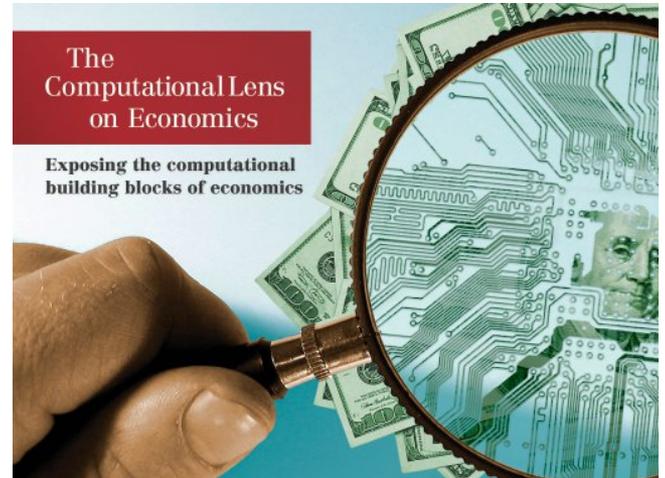
**Moral:** Even modest (10%) over-provisioning  
sufficient for near-optimal routing.



# The Computational Lens

## Classical model and results:

- the traffic model and its equilibria
  - [Wardrop 52];
  - [Beckman/McGuire/Winsten 56]
- inefficiency of equilibria [Pigou 1920]



## Contributions of the theory CS perspective:

- approximation as constructive compromise
  - lesson learned from coping with NP-completeness
- “price of anarchy” small in many cases
- identification of worst-case networks

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# F.C.C. Backs Proposal to Realign Airwaves

The New York Times

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September 28, 2012

By EDWARD WYATT

WASHINGTON — The government took a big step on Friday to aid the creation of new high-speed wireless Internet networks that could fuel the development of the next generation of smartphones and tablets, and devices that haven't even been thought of yet.

The five-member Federal Communications Commission unanimously approved a sweeping, though preliminary, proposal to reclaim public airwaves now used for broadcast television and auction them off for use in wireless broadband networks, with a portion of the proceeds paid to the broadcasters.

The initiative, which the F.C.C. said would be the first in which any government would pay to reclaim public airwaves with the intention of selling them, would help satisfy what many industry experts say is booming demand for wireless Internet capacity.

Mobile broadband traffic will increase more than thirtyfold by 2015, the commission estimates. Without additional airwaves to handle the traffic, officials say, consumers will face more dropped calls, connection delays and slower downloads of data.

# FCC: Buying Low, Selling High

## *Broadcast Television Incentive Auction (early 2016):*

- Reverse Auction: buy TV broadcast licenses
  - CBO estimate: \$15 billion cost
- Forward Auction: sell 4G wireless broadband licenses.
  - CBO estimate: \$40 billion revenue.
- Revenue to cover auction costs, fund a new first responder network, reduce the deficit (!)
  - “Middle Class Tax Relief and Job Creation Act”

# Bad Designs Cost Billions

## New Zealand, 1990:

- simultaneous sealed-bid 2<sup>nd</sup>-price auctions for 10 interchangeable TV broadcasting licenses
  - creates tricky coordination problem
- projected revenue: 250M; actual = 36M
- often huge difference between top two bids

US, 2009: 20 billion revenue.

# Reverse Auction Format

*“Descending Clock Auction”*:

[Milgrom-Segal 12]

- each round, each broadcaster offered a buyout price (decreases over time)
  - declined  $\Rightarrow$  exits, retains license
  - accepted  $\Rightarrow$  moves to next round
- different prices allowed for different broadcasters



Milgrom



Segal

# The Stopping Rule

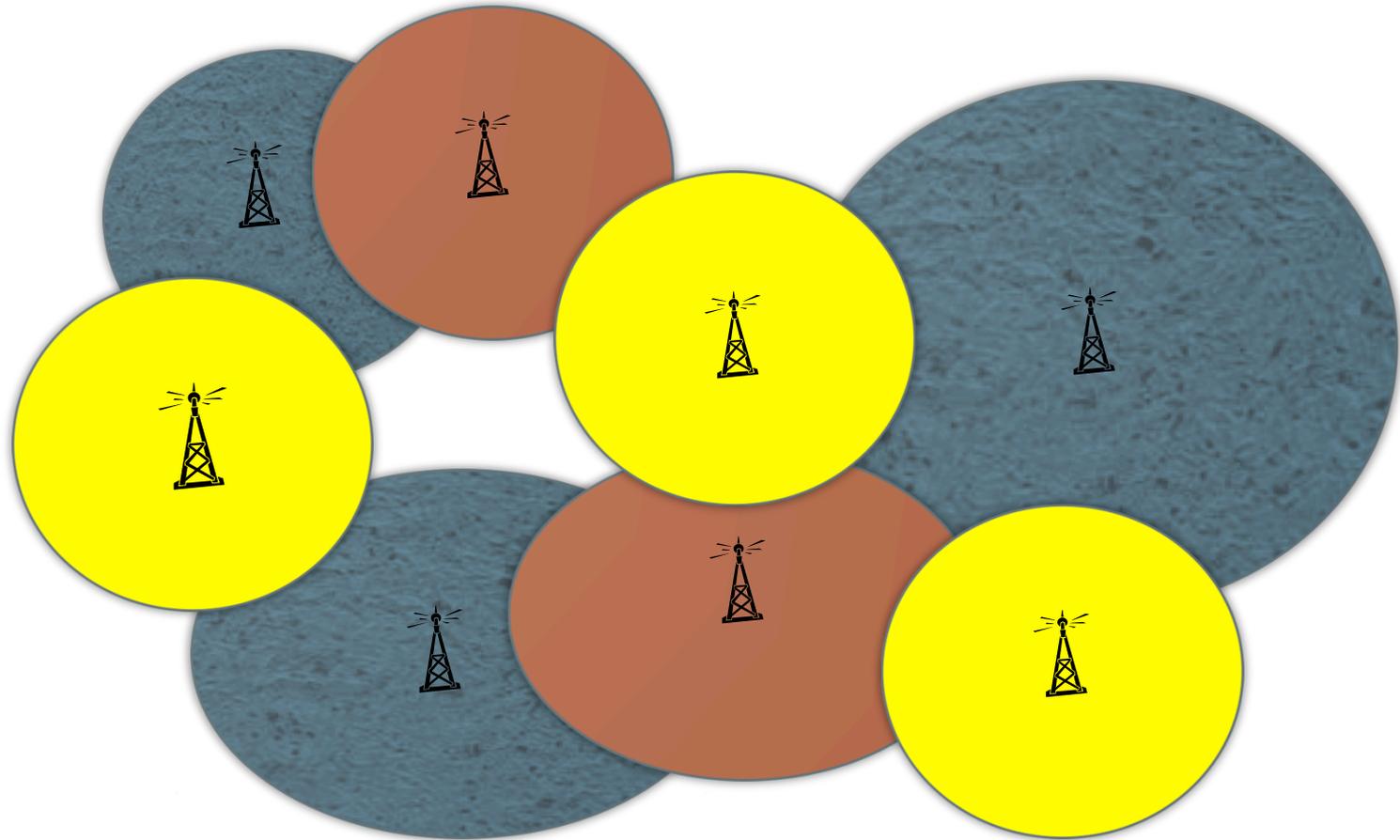
**Intuition:** stop auction when prices are as low as possible, subject to clearing enough spectrum.

**Example goal:** from channels 38-51, clear 10 of them nationwide.

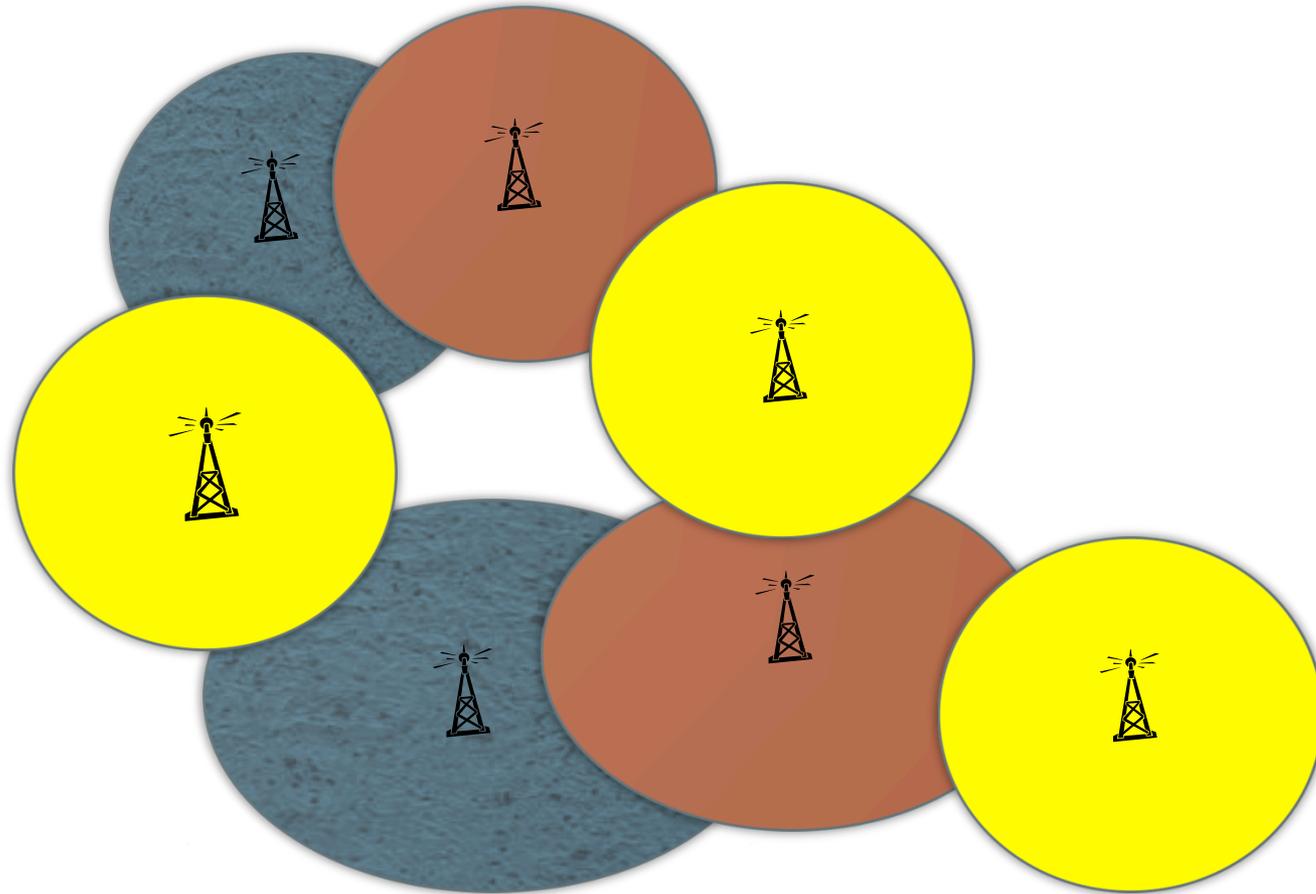
**Issue:** buyouts scattered across channels.

**Solution:** repack remaining TV stations into a smaller subset of channels (e.g., 38-41).

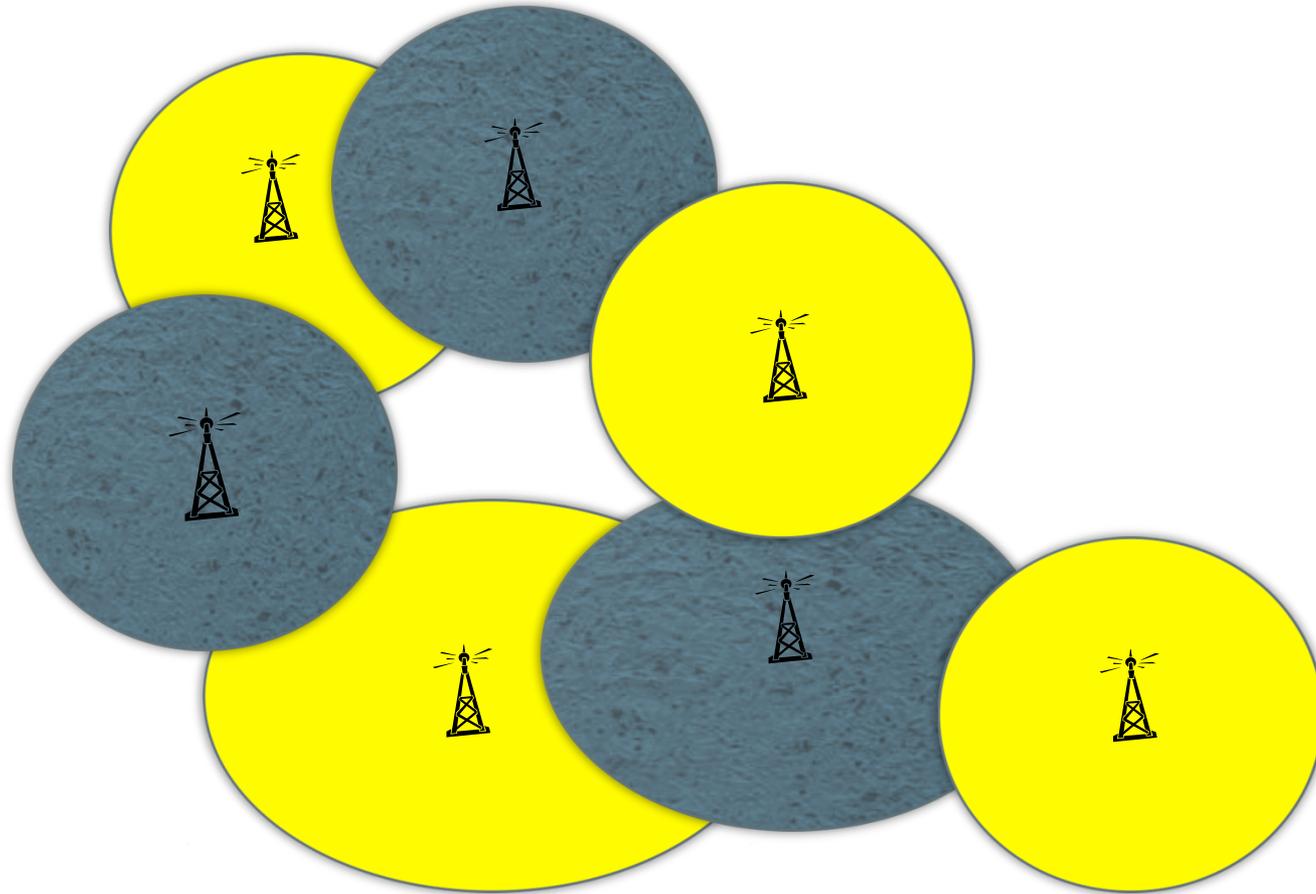
# The Repacking Problem



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# Influence of Theory CS

**Connection #1:** high-level auction format a generalization of

[Mehta/Roughgarden/Sundararajan 09]

- continues to enjoy the strong incentive guarantees of [MRS09]

**Connection #2:** proposed methods for discriminatory pricing related to

[Lehmann/O'Callaghan/Shoham 02]



Sundararajan



Shoham

# Influence of Theory CS

**Connection #3:** state-of-the-art algorithms for solving NP-complete problems necessary and sufficient to solve repacking problem quickly.

[Leyton-Brown et al. 13,14]

- encode as SAT
- use presolvers, solver configurations tuned to interference constraints



Leyton-Brown

# Forward Auction: Industry Rules of Thumb

**Folklore belief #1:** without strong complements, simple auctions work pretty well.

- loss in outcome quality (e.g., from “demand reduction”) appears small

**Folklore belief #2:** with strong complements, simple auctions aren’t good enough.

- loss in outcome quality can be big (e.g., because of the “exposure problem”)

# Corresponding Theory

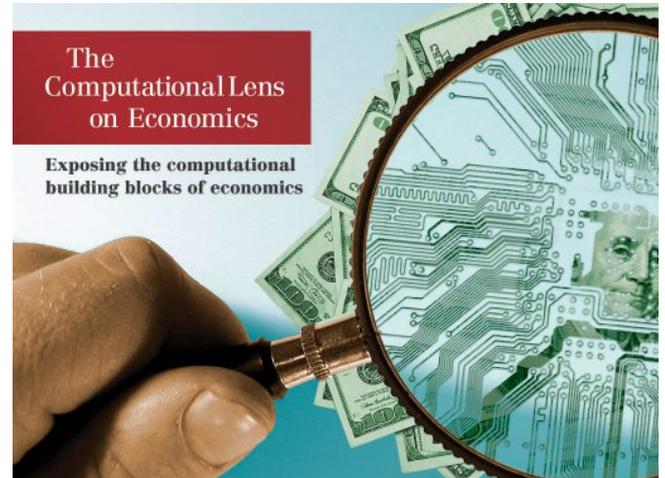
**Theorem 1:** without strong complements, simple auctions work pretty well.

[Christodoulou/Kovacs/Schapira 08], [Bhawalkar/Roughgarden 11,12], [Hassidim/Kaplan/Mansour/Nisan 11], [Feldman/Fu/Gravin/Lucier 13], [Syrkkanis/Tardos 13], [de Keijzer/Markakis/Schaefer/Telelis 13], [Duetting/Henzinger/Starnberger 13], [Babaioff/Lucier/Nisan/Paes Leme 13]

**Theorem 2:** [Dobzinski/Nisan/Schapira 05],[Roughgarden 14] with strong complements, simple auctions aren't good enough.

# The Computational Lens

- Reverse auction format builds on previous designs.
- Greedy heuristics for decreasing buyout prices.
- State-of-the-art SAT solvers needed for fast solution of repacking problems.
- Formalize rules of thumb for forward auction design using approximation (positive results) and communication complexity (negative results).



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# Constructive Nash's Theorem?

**Nash's Theorem (1950):** every finite game has at least one Nash equilibrium.

- many interpretations require the determination of an equilibrium
  - by the players, designer, etc.
- fixed-point proof offers no help



An equilibrium

**Challenge:** “more constructive” version.

- cf., “bounded rationality” [Simon]

# The Evidence Against

- Rabin (1957): “It is quite obvious that not all games considered in the theory can actually be played by human beings.”



Rabin

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- Rabin (1957): “It is quite obvious that not all games considered in the theory can actually be played by human beings.”
- Gilboa/Zemel (1989): Many problems about Nash equilibria are NP-complete.
- Dutta (2003): algebraic universality of Nash equilibria
- Hart/Mas-Colell (2003): no natural learning algorithms
- Savani/von Stengel (2004): Lemke-Howson can require exponential time to compute an equilibrium



Rabin

# Classifying the complexity of computing a Nash equilibrium

**Idea:** is computing a Nash equilibrium NP-complete?

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**Idea:** is computing a Nash equilibrium NP-complete?

**Answer:** [Megiddo 88]: probably not.

- reason: guaranteed existence

**Upshot:** Need to refine NP to obtain the right complexity class.

**Proposal:** [Papadimitriou 94] PPAD.



Papadimitriou

# Nash equilibria are intractable

**Theorem:** [Daskalakis/Goldberg/Papadimitriou 06, Chen/Deng/Teng 06] Computing a Nash equilibrium of a bimatrix game is PPAD-complete.

- also intractable in other senses [Etessami/Yannakakis 07, Hart/Mansour 09]

**Interpretation:** no general constructive version of Nash's theorem. Compromises are required.

- Tractable special cases, approximation, etc.

# The Computational Lens

- Nash's theorem gives universal existence
- Complexity theory shows impossibility of universal computational tractability
- Contributes to skepticism over universal predictive power of the Nash equilibrium



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# Conclusions

- many points of contact between theory CS and economics/game theory over past 15 years
  - including many not mentioned today
- many 21<sup>st</sup>-century computer science applications require economic reasoning
- the theory CS toolbox articulates computational barriers and offers constructive compromises