# CS364A: Exercise Set #6

Due by the beginning of class on Wednesday, November 6, 2013

**Instructions:**

(1) Turn in your solutions to all of the following exercises directly to one of the TAs (Kostas or Okke). Please type your solutions if possible and feel free to use the LaTeX template provided on the course home page. Email your solutions to `cs364a-aut1314-submissions@cs.stanford.edu`. If you prefer to hand-write your solutions, you can give it to one of the TAs in person at the start of the lecture.

(2) Your solutions will be graded on a "check/minus/zero" system, with "check" meaning satisfactory and "minus" meaning needs improvement.

(3) Solve these exercises and write up your solutions on your own. You may, however, discuss the exercises verbally at a high level with other students. You may also consult any books, papers, and Internet resources that you wish. And of course, you are encouraged to contact the course staff (via Piazza or office hours) to clarify the questions and the course material.

(4) No late assignments will be accepted.

## Lecture 11 Exercises

### Exercise 45

Consider a *multicommodity* network $G = (V, E)$, where for $i = 1, 2, \ldots, k$, $r_i > 0$ units of traffic travel from an origin $s_i \in V$ to a destination $t_i \in V$. Extend the definitions of a flow and of an equilibrium flow to multicommodity networks.

### Exercise 46

Extend the two expressions for total travel time to multicommodity networks.

### Exercise 47

Extend to multicommodity networks the result that every network with cost functions in a set $\mathcal{C}$ has POA at most the Pigou bound $\alpha(\mathcal{C})$.

### Exercise 48

Prove that if $\mathcal{C}$ is the set of cost functions of the form $c(x) = ax + b$ with $a, b \geq 0$, then the Pigou bound $\alpha(\mathcal{C})$ is $\frac{4}{3}$.

## Lecture 12 Exercises

### Exercise 49

Deduce from the first main result in lecture (i.e., that an equilibrium flow costs no more than an optimal flow with twice the traffic) the following statement.

Given a network $G = (V, E)$ and cost function $c_e$ for each edge $e \in E$, define modified cost functions by $\tilde{c}_e(x) = c_e(x/2)/2$ for each $e \in E$. If $\tilde{f}$ is an equilibrium flow in the network $G$ with the modified cost functions $\{\tilde{c}_e\}_{e \in E}$ and $f^*$ is an optimal flow in the network $G$ with the original cost functions $\{c_e\}_{e \in E}$, then the cost of $\tilde{f}$ (with respect to the modified cost functions) is no more than the cost of $f^*$ (with respect to the original cost functions).

Also, if $c_e(x) = 1/(u_e - x)$, then what is $\tilde{c}_e(x)$?

## Exercise 50

Prove that for every network and $\delta > 0$, the equilibrium flow has total travel time at most $\frac{1}{\delta}$ times that of an optimal flow that routes $(1 + \delta)$ times as much traffic.

## Exercise 51

Prove that the bound in the previous exercise is tight: for every $\delta, \epsilon > 0$ there is a network in which the equilibrium flow has total travel time at least $(\frac{1}{\delta} - \epsilon)$ times that of an optimal flow that routes $(1 + \delta)$ times as much traffic.

## Exercise 52

Extend to multicommodity networks the result that the cost of an equilibrium flow is at most that of an optimal flow that routes twice as much traffic (between each origin-destination pair).

## Exercise 53

Prove that for every pair $y, z \in \{0, 1, 2, \ldots\}$ of nonnegative integers,

$$y(z + 1) \leq \frac{5}{3}y^2 + \frac{1}{3}z^2.$$

## Exercise 54

The AAE example (Example 18.6 in the AGT book) shows that the POA of atomic selfish routing in networks with affine cost functions is at least 2.5. The example has four players. Give an example with three players that proves the same lower bound.