

CS264: Homework #6

Due by midnight on Wednesday, November 5, 2014

Instructions:

- (1) Students taking the course pass-fail should complete the exercises. Students taking the course for a letter grade should complete both the exercises and the problems.
- (2) All other instructions are the same as in previous problem sets.

Lecture 11 Exercises

Exercise 41

A *binary linear code* is a subset of $\{0, 1\}^n$ that arises as the kernel (over \mathbb{F}_2) of a matrix A — i.e., as the solutions to a system of the form $Ax = 0$.

Prove that the set of parity check codes described in class (for a fixed variable set V , ranging over all sets C and bipartite graphs $G = (V, C, E)$) is precisely the set of binary linear codes of length $|V|$.

Exercise 42

Recall the main theorem from lecture, for codes derived from graphs that satisfy the bounded-degree and expansion conditions: there is a constant $\delta > 0$ (independent of n) such that, if \mathbf{z} has Hamming distance less than $\delta n/2$ from the code word \mathbf{w} , then the unique optimal solution to the linear program (LP) is \mathbf{w} .

Prove that this statement holds in general if and only if it holds in the special where $\mathbf{w} = 0$.

Lecture 12 Exercises

Exercise 43

In this exercise, you can assume the following version of Hall's theorem: if $G = (A, B, E)$ is a bipartite graph with $|N(S)| \geq |S|$ for every $A \subseteq B$, then G has a matching in which all nodes of A are matched.

Prove the result needed for the proof in lecture: if $G = (A, B, E)$ is a bipartite graph such that, for a positive integer c , $|N(S)| \geq c|S|$ for every subset $S \subseteq A$, then there is a subset $F \subseteq E$ of edges such that (i) each vertex of B is incident to at most one edge of F ; and (ii) each vertex of A is incident to at least c edges of F .

Problems

Problem 20

(15 points) This problem shows how to use the probabilistic method to prove that good bipartite expanders exist. Let d be a positive integer (a constant). Consider vertex sets A and B , with $|A| = n$ and $|B| = c|A|$ for a constant $c \in (0, 1)$. Obtain a random graph G by choosing, independently for each $a \in A$, d neighbors (with replacement) uniformly at random from B . With probability 1, G is a bipartite graph in which all vertices of A have degree d .¹

¹In lecture we also insisted that the right-hand side vertices have bounded degree. For simplicity, we drop this constraint for this problem.

Prove that there is a constant $\delta > 0$, which can depend on c and/or d but not on n , such that, with probability approaching 1 as $n \rightarrow \infty$,

$$|N(S)| \geq \frac{3}{4}d|S|$$

for every set $S \subseteq A$ with $|S| \leq \delta n$.

[Hint: Use the usual maneuvers from randomized algorithms, like the Chernoff and Union bounds. For example, you could consider the probability that a vertex of $N(S)$ has unique neighbor in S .]

Problem 21

This problem gives a simple and more practical alternative decoding algorithm to the one given in Lectures #11 and #12.² Given a corrupted code word \mathbf{z}_0 , the *SS algorithm* does the following:

(SS) While there is at least one variable $i \in V$ such that more than half of the parity checks in $N(i)$ are unsatisfied, modify \mathbf{z} by flipping the value of an arbitrary such variable.

(a) (5 points) Prove that, no matter what the bipartite graph G (defining the parity check code) and initial corrupted word \mathbf{z}_0 are, the SS algorithm is guaranteed to terminate in polynomial time.

(b) (8 points) For this and all remaining parts, assume that the graph G defining the code satisfies the first and third conditions described in lecture (the V -side is d -regular, and the expansion condition).

Prove that, if the current solution \mathbf{z} differs from a code word \mathbf{w} in $m \leq \delta n$ variables, then there are more than $dm/2$ edges between currently corrupted variables (i.e., variables on which \mathbf{w} and \mathbf{z} differ) and currently unsatisfied parity checks. Here δ denotes the same constant as in the expansion condition stated in lecture.

[Hint: In addition to the expansion condition, use the fact that a parity check j containing a corrupted variable can only be satisfied if it contains at least two corrupted variables.]

(c) (3 points) Explain why (b) implies that, if the current Hamming distance from \mathbf{z} to the nearest code word is at most δn , then the SS algorithm will flip the value of some variable.

(d) (2 points) Discuss why (c) does not necessarily imply that the Hamming distance between \mathbf{z} and the nearest code word strictly decreases with the number of iterations of the algorithm.

(e) (5 points) Prove that, if the initial code word \mathbf{z}_0 has Hamming distance at most $\delta n/2$ from the nearest code word, then in every iteration of the SS algorithm, the current solution \mathbf{z} has Hamming distance less than δn from the nearest code word.

[Hint: use (b) and a monotonicity argument familiar from part (a).]

(f) (2 points) Conclude that if the SS algorithm is initialized with a corrupted code word \mathbf{z}_0 with Hamming distance at most $\delta n/2$ from the nearest code word \mathbf{w} , then the SS algorithm is guaranteed to terminate with the solution \mathbf{w} .

²There are, however, some parameter ranges where LP decoding is known to work and this simpler algorithm is not known to work.