COMS 4995 (Randomized Algorithms): Exercise Set #10

For the week of November 18–22, 2019

Instructions:

(1) *Do not turn anything in.*

(2) The course staff is happy to discuss the solutions of these exercises with you in office hours or in the course discussion forum.

(3) While these exercises are certainly not trivial, you should be able to complete them on your own (perhaps after consulting with the course staff or a friend for hints).

Exercise 46

Consider a random walk on a path graph (with vertex set $\{1, 2, \ldots, n\}$), started at vertex 2. What is the probability that the random walk visits vertex n before ever visiting vertex 1?[1]

[Hint: Formulate a recurrence for the analogous probability when the random walk starts at the vertex i.]

Exercise 47

Prove that a d-regular undirected graph $G = (V, E)$ is connected if and only if the second-largest eigenvalue $\lambda_2(A)$ of its adjacency matrix A is less than d.

Exercise 48

Prove that a d-regular and connected undirected graph $G = (V, E)$ is not bipartite if and only if the smallest (i.e., most negative) eigenvalue $\lambda_n(A)$ of its adjacency matrix A is greater than $-d$.

Exercise 49

Consider a connected and d-regular undirected graph $G = (V, E)$. Consider a random walk on G, with the extra twist that at each time step there is a 50% chance of stalling (i.e., remaining at the same vertex for the next time step).

(a) What is the transition matrix of the corresponding Markov chain?

(b) Prove that all of the eigenvalues of this matrix are nonnegative.

Exercise 50

Recall that the Laplacian matrix $L(G)$ of a d-regular undirected graph $G = (V, E)$ is defined as $dI - A$, where I is the identity matrix and A is G's adjacency matrix.

(a) In Lecture #20 we gave a combinatorial interpretation of the adjacency matrix as an operator (replacing one labeling of the vertices with another). Give a combinatorial interpretation of $L(G)$ as an operator.

[1] Recall this problem was relevant in Lecture #19 when arguing a lower bound of $\Omega(n^3)$ on the cover time of the lollipop graph.

(b) We mentioned briefly in lecture that the quadratic form $x^\top Mx$ is important for bounding the eigenvalues of a symmetric matrix M. Give a simple formula for the quadratic form $x^\top L(G)x$ corresponding to a Laplacian matrix, along with a combinatorial interpretation.