

# CS364B: Exercise Set #2

Due by the beginning of class on Wednesday, January 22, 2014

## Instructions:

- (1) Turn in your solutions to all of the following exercises directly to the TA (Okke). Please type your solutions if possible and feel free to use the LaTeX template provided on the course home page. Email your solutions to `cs364b-win1314-submissions@cs.stanford.edu`. If you prefer to hand-write your solutions, you can give it to Okke in person at the start of the lecture.
- (2) Your solutions will be graded on a “check/minus/zero” system, with “check” meaning satisfactory and “minus” meaning needs improvement.
- (3) Solve these exercises and write up your solutions on your own. You may, however, discuss the exercises verbally at a high level with other students. You may also consult any books, papers, and Internet resources that you wish. And of course, you are encouraged to contact the course staff (via Piazza or office hours) to clarify the questions and the course material.
- (4) No late assignments will be accepted.

## Lecture 3 Exercises

### Exercise 9

These 5 exercises walk through a proof of Lemma 3. Consider scenario #3. Let  $(\mathbf{p}, M^*)$  be the VCG outcome. Let  $(\mathbf{q}, M)$  be an arbitrary  $\epsilon$ -Walrasian equilibrium (recall the definition from Exercise #5 from the last exercise set).

For this exercise, assume that there is a good  $j_1$  with  $q(j_1) < p(j_1) - \epsilon\ell$ , where  $\ell$  is a positive integer. Observe that this implies that  $p(j_1) > 0$ . Argue that at the prices  $\mathbf{p}$ , there are at least two bidders  $i_1, i_2$  for which  $j_1$  is a favorite good (i.e.,  $j_1 \in D_i(\mathbf{p}) := \operatorname{argmax}_{U \cup \emptyset} \{v_i(j_1) - p(j_1)\}$ ).

[Hint: recall that  $\mathbf{p}$  is the smallest Walrasian price vector.]

### Exercise 10

Continuing the previous exercise, prove that  $i_1, i_2$  are both matched (to items of  $U$ ) in  $M$ .

[Hint: use that  $q(j_1) < p(j_1) - \epsilon$  and that  $(q, M)$  is an  $\epsilon$ -Walrasian equilibrium.]

### Exercise 11

Continuing the previous exercise, the key claim is that, for  $k = 1, 2, \dots, \ell$ , there is a set  $U_k$  of  $k$  goods and a set  $B_k$  of  $k + 1$  bidders such that:

- (1) for every  $j \in U_k$ ,  $q(j) < p(j) - \epsilon(\ell - k + 1)$ ;
- (2) for every  $i \in B_k$ , at the prices  $\mathbf{p}$ ,  $i$  has a favorite item (i.e., an item of  $D_i(\mathbf{p})$ ) in  $U_k$ ;
- (3) every bidder of  $B_k$  is matched in  $M$ .

Observe that the previous two exercises establish the base case with  $U_1 = \{j_1\}$  and  $B_1 = \{i_1, i_2\}$ . Observe also that, if we prove this, then since property (3) implies that  $B_k$  can't be larger than  $\min\{m, n\}$ , we'll have the desired result: every item price  $q(j)$  in an  $\epsilon$ -Walrasian equilibrium price vector is at most  $\epsilon \cdot \min\{n, m\}$  less than the VCG price  $p(j)$ .

Now for the exercise. For the inductive step, consider sets  $U_{k-1}, B_{k-1}$  that satisfy (1)–(3). Property (3) and the fact that  $|U_{k-1}| < |B_{k-1}|$  imply that there is a bidder  $i \in B_{k-1}$  that is matched to an item  $j_k \notin U_{k-1}$  in  $M$ . Prove that  $p(j_k) - q(j_k) > \epsilon(\ell - k + 1)$ , and hence we can define  $U_k = U_{k-1} \cup \{j_k\}$  and satisfy property (1).

[Hint: use that  $(\mathbf{p}, M^*)$  is a WE and that  $(\mathbf{q}, M)$  is an  $\epsilon$ -WE.]

### Exercise 12

Continuing the previous exercise, prove that there is a bidder  $i_{k+1} \notin B_{k-1}$  such that  $D_{i_{k+1}}(\mathbf{p})$  includes a good of  $U_k$ . Hence, we can define  $B_k = B_{k-1} \cup \{i_{k+1}\}$  and satisfy property (2).

[Hint: this generalizes Exercise 1.]

### Exercise 13

Continuing the previous exercise, prove that  $B_k$  satisfies property (3), which completes the inductive step.

[Hint: this is similar to Exercise 2.]

## Lecture 4 Exercises

### Exercise 14

Construct an iterative auction  $\mathcal{A}$  with the following properties:

- (i) Sincere bidding always yields a VCG outcome (or a close approximation of it).
- (ii) Sincere bidding is not (a close approximation of) an EPNE.

[I.e., an auction that iteratively computes the VCG outcome with sincere bidding is not automatically EPIC.]

[Hint: It should be enough to consider 3 goods, 2 unit-demand bidders, and a modification of the Crawford-Knoer auction.]

### Exercise 15

Consider  $m$  identical items and  $n$  bidders with downward-sloping valuations (scenario #4). Define a *Walrasian equilibrium* as a price  $q$  and an allocation  $n$ -vector  $\mathbf{x}$  such that:

1.  $\sum_{i=1}^n x_i \leq m$ ;
2. for every bidder  $i$ ,  $x_i \in \operatorname{argmax}\{\sum_{j=1}^k v_i(j) - q \cdot k\}_{j=0,1,\dots,m}$ ;
3.  $\sum_{i=1}^n x_i < m$  only if  $q = 0$ .

Prove that there is always a Walrasian equilibrium.

[Given existence, there is clearly a smallest Walrasian price.]

### Exercise 16

Continuing the previous exercise, prove that, for every bidder  $i$ ,  $i$ 's utility in the (truthful) VCG outcome is at least as large as in every Walrasian equilibrium.

[This is the analog of our result in scenario #3 (unit-demand bidders) that VCG payments are bounded above by every Walrasian price vector.]

## Exercise 17

Consider the following direct-revelation mechanism for scenario #4:

1. Collect a downward-sloping bid  $\{b_i(j)\}_{j=1,2,\dots,m}$  from each bidder  $i$ .
2. Treating the bids as valuations, output the smallest Walrasian equilibrium  $(q, \mathbf{x})$ .

Show by example that this mechanism is not DSIC.

[Hint: there are simple examples where a bidder wants to engage in “demand reduction,” meaning accepting a smaller number of units than with truthful bidding, in order to get them at a much cheaper price.]