CS261: Exercise Set #5

For the week of February 1–5, 2016

Instructions:

(1) *Do not turn anything in.*

(2) The course staff is happy to discuss the solutions of these exercises with you in office hours or on Piazza.

(3) While these exercises are certainly not trivial, you should be able to complete them on your own (perhaps after consulting with the course staff or a friend for hints).

Exercise 21

Consider the following linear programming relaxation of the maximum-cardinality matching problem:

$$\max \sum_{e \in E} x_e$$

subject to

$$\sum_{e \in \delta(v)} x_e \leq 1 \qquad \text{for all } v \in V$$

$$x_e \geq 0 \qquad \text{for all } e \in E,$$

where $\delta(v)$ denotes the set of edges incident to vertex v.

We know from Lecture #9 that for bipartite graphs, this linear program always has an optimal 0-1 solution. Is this also true for non-bipartite graphs?

Exercise 22

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^m$ be a set of n m-vectors. Define C as the *cone* of $\mathbf{x}_1, \ldots, \mathbf{x}_n$, meaning all linear combinations of the $\mathbf{x}_i$'s that use only nonnegative coefficients:

$$C = \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{x}_i \, : \, \lambda_1, \ldots, \lambda_n \geq 0 \right\}.$$

Suppose $\alpha \in \mathbb{R}^m$, $\beta \in \mathbb{R}$ define a "valid inequality" for C, meaning that

$$\alpha^T \mathbf{x} \geq \beta$$

for every $\mathbf{x} \in C$. Prove that

$$\alpha^T \mathbf{x} \geq 0$$

for every $\mathbf{x} \in C$, so α and 0 also define a valid inequality.

[Hint: Show that $\beta > 0$ is impossible. Then use the fact that if $\mathbf{x} \in C$ then $\lambda \mathbf{x} \in C$ for all scalars $\lambda \geq 0$.]

Exercise 23

Verify that the two linear programs discussed in the proof of the minimax theorem (Lecture #10),

$$\max v$$

subject to

$$\begin{aligned} v - \sum_{i=1}^{m} a_{ij} x_i &\le 0 \qquad \text{for all } j = 1, \ldots, n \\ \sum_{i=1}^{m} x_i &= 1 \\ x_i &\ge 0 \qquad \text{for all } i = 1, \ldots, m \\ v &\in \mathbb{R}, \end{aligned}$$

and

$$\min w$$

subject to

$$\begin{aligned} w - \sum_{j=1}^{n} a_{ij} y_j &\ge 0 \qquad \text{for all } i = 1, \ldots, m \\ \sum_{j=1}^{n} y_j &= 1 \\ y_j &\ge 0 \qquad \text{for all } j = 1, \ldots, n \\ w &\in \mathbb{R}, \end{aligned}$$

are both feasible and are dual linear programs. (As in lecture, $\mathbf{A}$ is an $m \times n$ matrix, with a_{ij} specifying the payoff of the row player and the negative of the payoff of the column player when the former chooses row i and the latter chooses column j.)

Exercise 24

Consider a linear program with n decision variables, and a feasible solution $\mathbf{x} \in \mathbb{R}^n$ at which less than n of the constraints hold with equality (i.e., the rest of the constraints hold as strict inequalities).

(a) Prove that there is a direction $\mathbf{y} \in \mathbb{R}^n$ such that, for all sufficiently small $\epsilon > 0$, $\mathbf{x} + \epsilon\mathbf{y}$ and $\mathbf{x} - \epsilon\mathbf{y}$ are both feasible.

(b) Prove that at least one of $\mathbf{x} + \epsilon\mathbf{y}, \mathbf{x} - \epsilon\mathbf{y}$ has objective function value at least as good as $\mathbf{x}$.

[Context: these are the two observations that drive the fact that a linear program with a bounded feasible region always has an optimal solution at a vertex. Do you see why?]

Exercise 25

Recall from Problem #12(e) (in Problem Set #2) the following linear programming formulation of the s-t shortest path problem:

$$\min \sum_{e \in E} c_e x_e$$

subject to

$$\sum_{e \in \delta^+(S)} x_e \geq 1 \qquad \text{for all } S \subseteq V \text{ with } s \in S,\ t \notin S$$

$$x_e \geq 0 \qquad \text{for all } e \in E.$$

Prove that this linear program, while having exponentially many constraints, admits a polynomial-time separation oracle (in the sense of the ellipsoid method, see Lecture #10).