CS261: Exercise Set #9

For the week of February 29–March 4, 2016

Instructions:

(1) *Do not turn anything in.*

(2) The course staff is happy to discuss the solutions of these exercises with you in office hours or on Piazza.

(3) While these exercises are certainly not trivial, you should be able to complete them on your own (perhaps after consulting with the course staff or a friend for hints).

Exercise 41

Recall the Vertex Cover problem from Lecture #17: the input is an undirected graph $G = (V, E)$ and a non-negative cost c_v for each vertex $v \in V$. The goal is to compute a minimum-cost subset $S \subseteq V$ that includes at least one endpoint of each edge.

The natural greedy algorithm is:

- $S = \emptyset$
- while S is not a vertex cover:
  - add to S the vertex v minimizing (c_v/# newly covered edges)
- return S

Prove that this algorithm is not a constant-factor approximation algorithm for the vertex cover problem.

Exercise 42

Recall from Lecture #17 our linear programming relaxation of the Vertex Cover problem (with nonnegative edge costs):

$$\min \sum_{v \in V} c_v x_v$$

subject to

$$x_v + x_w \geq 1 \qquad \text{for all edges } e = (v, w) \in E$$

and

$$x_v \geq 0 \qquad \text{for all vertices } v \in V.$$

Prove that there is always a *half-integral* optimal solution $\mathbf{x}^*$ of this linear program, meaning that $x_v^* \in \{0, \frac{1}{2}, 1\}$ for every $v \in V$.

[Hint: start from an arbitrary feasible solution and show how to make it "closer to half-integral" while only improving the objective function value.]

Exercise 43

Recall the primal-dual algorithm for the vertex cover problem — in Lecture #17, we showed that this is a 2-approximation algorithm. Show that, for every constant $c < 2$, there is an instance of the vertex cover problem such that this algorithm returns a vertex cover with cost more than c times that of an optimal vertex cover.

Exercise 44

Prove *Markov's inequality*: if X is a non-negative random variable with finite expectation and $c > 1$, then

$$\mathbf{Pr}[X \geq c \cdot \mathbf{E}[X]] \leq \frac{1}{c}.$$

Exercise 45

Let X be a random variable with finite expectation and variance; recall that $\mathrm{Var}[X] = \mathbf{E}\big[(X - \mathbf{E}[X])^2\big]$ and $\mathrm{StdDev}[X] = \sqrt{\mathrm{Var}[X]}$. Prove *Chebyshev's inequality*: for every $t > 1$,

$$\mathbf{Pr}[|X - \mathbf{E}[X]\,| \geq t \cdot \mathrm{StdDev}[X]] \leq \frac{1}{t^2}.$$

[Hint: apply Markov's inequality to the (non-negative!) random variable $(X - \mathbf{E}[X])^2$.]